XL. *Methodus inveniendi Lineas Curvas ex proprietatibus Variationis Curvaturæ. Auctore* Nicolao Landerbeck, *Mathes. Profess. in Acad. Upsaliensi Adjuncto. Communicated by* Nevil Maskelyne, *D. D. F. R. S. and Astronomer Royal.*

Read July 1, 1784.

PARS SECUNDA*.

CURVAS, ex proprietate variationis curvaturæ invenire, indice per functionem coordinatarum cujusdam expresso, problema etsi indeterminatum est; juvat tamen ad curvas cognoscendas, quum facile et sponte sese offerunt conditiones determinantes qui rei conveniunt et quæ in casu quovis examini subjecto locum habent. Quo consilio et qua arte calculum inire oporteat, ut et hæc et his affinia peragenda sint, quæ ad curvas ex curvaturæ variatione cognoscendas pertineant, per theoremata quæ sequuntur, exponere conabor.

THEOREMA I. (Vide tab. XXI. fig. 2.)

Si curvæ cujusdam LC index variationis curvaturæ sit T, radius curvedinis R, sinus anguli BCD p, posito sinu toto 1, arcus curvæ LC z coordinatæ perpendiculares x et y earumque fluxiones dp, dz, dx, et dy dicantur, erit $\frac{dz}{\int Tdz} = -\frac{dp}{\sqrt{1-p^2}}$.

Quoniam $dx = -Rdp$ et $dz = -\frac{dx}{\sqrt{1-p^2}}$ habetur $\frac{dz}{R} = -\frac{dp}{\sqrt{1-p^2}}$

* See Vol. LXXIII. p. 456.

 et

et quum $d\mathrm{R}=\mathrm{T}dz$ erit $\mathrm{R}=\int \mathrm{T}dz$ et substitutione $\frac{dz}{\int \mathrm{T}dz} = -\frac{dp}{\sqrt{1-p^2}}$.

Cor. 1. Hinc obtinetur $\frac{dx}{\mathrm{R}} = -dp$, $\frac{dy}{\mathrm{R}} = -\frac{p\,dp}{\sqrt{1-p^2}}$ et $\frac{dz}{\mathrm{R}} = -\frac{dp}{\sqrt{1-p^2}}$.

Cor. 2. Si Tangens anguli BCD per r, Secans per s designentur habetur $\frac{dz}{\int \mathrm{T}dz} = -\frac{dr}{1+r^2}$ et $\frac{dz}{\int \mathrm{T}dz} = -\frac{ds}{s\sqrt{s^2-1}}$.

Schol. 1. Ex hoc theoremate facilis deducitur methodus generaliter calculandi variationem curvaturæ curvæ cujuscumque. Nam $\int \mathrm{T}dz = -\frac{dz\sqrt{1-p^2}}{dp}$, quantitas vero $\frac{dz\sqrt{1-p^2}}{dp}$ datur, data inter x et y relatione. Sit valor quantitatis $-\frac{dz\sqrt{1-p^2}}{dp} = \mathrm{Z}$ functioni curvæ z, $\int \mathrm{T}dz = \mathrm{Z}$ et sumtis fluxionibus $\mathrm{T}dz = \overset{1}{\mathrm{Z}}dz$ qua $\mathrm{T} = \overset{1}{\mathrm{Z}}$ functioni ipsius z. Si valor quantitatis $-\frac{dz\sqrt{1-p^2}}{dp} = \mathrm{P}$ per p expressus, erit $\int \mathrm{T}dz = \mathrm{P}$ sumtisque fluxionibus $\mathrm{T}dz = \overset{1}{\mathrm{P}}dp$ et $\mathrm{T} = \frac{\overset{1}{\mathrm{P}}dp}{dz}$, quæ functio est quantitatis p, in potestate semper est $\frac{dp}{dz}$ per p exprimere.

Schol. 2. Hujus etiam theorematis subsidio inveniri possunt curvæ ex data relatione inter T et z, R et z, R et y, et R et p. Si enim sit $\mathrm{T}=\mathrm{Z}$ functioni quantitatis z, erit $\int \mathrm{T}dz = \int \mathrm{Z}dz + \mathrm{A}$, vi theorematis $\frac{dz}{\int \mathrm{Z}dz+\mathrm{A}} \left(= \frac{dz}{\int \mathrm{T}dz}\right) = -\frac{dp}{\sqrt{1-p^2}}$ et integratione $\int \frac{dz}{\int \mathrm{Z}dz+\mathrm{A}} + \mathrm{C} = -\frac{dp}{\sqrt{1-p^2}}$. Posita $\int \frac{dz}{\int \mathrm{Z}dz+\mathrm{A}} + \mathrm{C} = b$ et N numerus

merus cujus logarithmus hyperbolicus 1 habetur $\sqrt{1-p^2} = \frac{N^{b\sqrt{-1}} - N^{-b\sqrt{-1}}}{2\sqrt{-1}}$ et $p = \frac{N^{b\sqrt{-1}} + N^{-b\sqrt{-1}}}{2}$, quæ functiones ſunt quantitatis z, quibus poſitis $\overset{|}{Z}$ et $\sqrt{1-\overset{|}{Z}^2}$ reſpective proveniunt $x (= \int dz \sqrt{1-p^2}) = \int \overset{|}{Z} dz$ et $y (= \int p dz) = \int dz \sqrt{1-\overset{|}{Z}^2}$ quarum alterutra curvarum indoles innoteſcit.

Si $R = X$ functioni abſciſſæ x provenit $\frac{dx}{X} \left(= \frac{dx}{R}\right) = -dp$ et integratione $\overset{|}{X} \left(= C - \int \frac{dx}{R}\right) = p$ unde $\sqrt{1-p^2} = \sqrt{1-\overset{|}{X}^2}$ et $y \left(= \int \frac{p dx}{\sqrt{1-p^2}}\right) = \int \frac{\overset{|}{X} dx}{\sqrt{1-\overset{|}{X}^2}}$ æquatio curvæ indolem exprimens.

Et ſi $R = Y$ functioni ordinatæ y, habetur $\frac{dy}{Y} \left(= \frac{dy}{R}\right) = -\frac{p dp}{\sqrt{1-p^2}}$ et integratione $\overset{|}{Y} \left(= \int \frac{dy}{Y} + C\right) = \sqrt{1-p^2}$, unde $p = \sqrt{1-\overset{|}{Y}^2}$ et $x \left(= \int \frac{dy\sqrt{1-p^2}}{p}\right) = \int \frac{\overset{|}{Y} dy}{\sqrt{1-\overset{|}{Y}^2}}$ quæ exprimit naturam curvæ.

Hinc colligitur quod quoties $T dz$ perfecte integretur et $\int \frac{dz}{\int Z dz + A}$ obtineatur per arcus circulares dum aut $\int \overset{|}{Z} dz$ aut $\int dz \sqrt{1-\overset{|}{Z}^2}$ abſolutam admittat integrationem, curvæ erunt rectificabiles, et algebraicæ, ſi relatio inter x et z vel inter y et z in relationem algebraicam inter x et y permutari poſſit.

Evidens etiam eſt quod ſi X functio eſt algebraica quantitatis x; vel Y quantitatis y, et non ſolum $\frac{dx}{X}$ vel $\frac{dy}{Y}$ ſed etiam $\frac{\overset{|}{X} dx}{\sqrt{1-\overset{|}{X}^2}}$ vel $\frac{\overset{|}{Y} dy}{\sqrt{1-\overset{|}{Y}^2}}$ quantitates perfecte integrabiles, curvæ evadunt algebraicæ, alias tranſcendentes.

Exempl.

Exempl. 1. Invenienda sit curva ubi variatio curvaturæ $T = \frac{3 \cdot \overline{8a+27z}|^{\frac{2}{3}} - 2a^{\frac{2}{3}}}{a^{\frac{1}{3}} \sqrt{\overline{8a+27z}|^{\frac{2}{3}} - 4a^{\frac{2}{3}}}}$. Ut simplicior reddatur calculus ponatur $\overline{8a+27z}|^{\frac{2}{3}} = u$ et $a^{\frac{2}{3}} = b$ erit $z = \frac{u^{\frac{3}{2}} - 8b^{\frac{3}{2}}}{27}$, $dz = \frac{du\sqrt{u}}{18}$, $T = \frac{3u-2b}{\sqrt{b}\sqrt{u-4b}}$ et $\int T dz = \frac{u\sqrt{u}\sqrt{u-4b}}{18\sqrt{b}} + A$; sit constans hæc $A = a$, quod accidit evanescente $\int T dz\, u = b$, habetur per theorema $\frac{du\sqrt{b}}{u\sqrt{u-4b}} \left(= \frac{dz}{\int T dz}\right) = -\frac{dp}{\sqrt{1-p^2}}$ et integratione $\int \frac{du\sqrt{b}}{u\sqrt{u-4b}} + C = -\int \frac{dp}{\sqrt{1-p^2}}$, cujus æquationis termini quum sint arcus circulares quorum sinus $\sqrt{1-p^2} = \frac{\sqrt{u-4b}}{\sqrt{u}}$ et cosinus $p = \frac{2\sqrt{b}}{\sqrt{u}}$, posito arcu constanti $C = 0$, obtinetur $y \left(= \int p dz\right) = \int \frac{du\sqrt{b}}{9} + B) = \frac{\sqrt{u-4b}\,\sqrt{b}}{9}$ nam $B = \frac{4b\sqrt{b}}{9}$, posita $y = 0$ et $u = 4b$, atque $x\left(= \int dz\sqrt{1-p^2}\right) = \int \frac{du\sqrt{u-4b}}{18} = \frac{\overline{u-4b}|^{\frac{3}{2}}}{27}$ quibus æquationibus exterminata u et substituta a habetur $y^3 = ax^2$ æquatio pro parabola cubica.

Exempl. 2. Si sit variatio curvaturæ $T = \frac{2z}{a}$ erit $\int T dz \left(= \int \frac{2z dz}{a}\right) = \frac{z^2}{a} + A$ et si $Z = 0$ $\int T dz = a$ erit constans $A = a$, atque vi theorematis $\frac{a dz}{a^2+z^2} \left(= \frac{dz}{\int T dz}\right) = -\frac{dp}{\sqrt{1-p^2}}$ et integratione $\int \frac{a dz}{a^2+z^2} + C = -\int \frac{dp}{\sqrt{1-p^2}}$; posito arcu constanti $C = 0$ cæteri sunt æquales eorumque sinus et cosinus, unde $\sqrt{1-p^2} = \frac{z}{\sqrt{a^2+z^2}}$, $p = \frac{a}{\sqrt{a^2+z^2}}$ et $dx \left(= dz\sqrt{1-p^2}\right) = \frac{z dz}{\sqrt{a^2+z^2}}$ et dy (=

 $p dz$)

$pdz) = \frac{adz}{\sqrt{a^2+z^2}}$, quibus conſtat curvam eſſe catenariam.

Exempl. 3. Sit variatio curvaturæ $T = \frac{a-z}{\sqrt{2az-z^2}}$, evadit $\int Tdz = \sqrt{2az-z^2}$, per theorema $\frac{dz}{\sqrt{2az-z^2}} \left(= \frac{dz}{\int Tdz}\right) = -\frac{dp}{\sqrt{1-p^2}}$ et per integrationem $\int \frac{dz}{\sqrt{2az-z^2}} + C = -\int \frac{dp}{\sqrt{1-p^2}}$, ſi arcus ille conſtans $C = 0$, cæteri ſunt æquales eorumque ſinus et coſinus, quo $\sqrt{1-p^2} = \frac{\sqrt{2ax-x^2}}{a}$, $p = \frac{a-z}{a}$ et $y \left(= \int pdz\right) = \int \frac{a-zdz}{a}\Big) = \frac{2az-z^2}{a}$ æquatio pro cycloide ordinaria.

THEOREMA II.

Manentibus antea adhibitis denominationibus erit $\frac{dx}{y+\int Tdx} = -\frac{dp}{\sqrt{1-p^2}}$.

Quoniam $\frac{dx}{R} = -dp$, erit dividendo per $\sqrt{1-p^2}$, $\frac{dx}{R\sqrt{1-p^2}} = -\frac{dp}{\sqrt{1-p^2}}$. Propter $1 : \sqrt{1-p^2} :: CD(R) : CF = R\sqrt{1-p^2}$, ſed $dz : dx :: Tdz : Tdx$, quæ fluxio eſt ipſius DE, quare $DE = \int Tdx$, unde $CF = y + \int Tdx$ qua pro $R\sqrt{1-p^2}$ ſubſtituta, prodit $\frac{dx}{y+\int Tdx} = -\frac{dp}{\sqrt{1-p^2}}$.

Cor. 1. Quantitas $dy + Tdx$ ſemper eſt perfecte integrabilis. Nam $Tdx = -\frac{ddx\sqrt{1-p^2}}{dp}$ et $dy = \frac{pdx}{\sqrt{1-p^2}}$ unde $dy + Tdx = \frac{pdx}{\sqrt{1-p^2}} - \frac{ddx\sqrt{1-p^2}}{dp}$ et integratione $y + \int Tdx = -\frac{dx\sqrt{1-p^2}}{dp}$.

Cor.

Cor. 2. Dicatur ſemichorda curvaturæ CF F, obtinetur $\frac{dx}{F} = -\frac{dp}{\sqrt{1-p^2}}$, $\frac{dy}{F} = -\frac{pdp}{1-p^2}$ et $\frac{dz}{F} = -\frac{dp}{1-p^2}$.

Cor. 3. Si Tangens anguli BCD per r, Secans per s deſignentur habetur $\frac{dx}{y+\int Tdx} = -\frac{dr}{1+r^2}$ et $\frac{dx}{y+\int Tdx} = -\frac{ds}{s\sqrt{s^2-1}}$.

Schol. 1. Per hoc theorema via etiam patet calculandi generaliter variationem curvaturæ. Eſt enim $y+\int Tdx = -\frac{dx\sqrt{1-p^2}}{dp}$, quantitas vero $\frac{dx\sqrt{1-p^2}}{dp}$ datur data inter x et p relatione. Sit valor quantitatis $-\frac{dx\sqrt{1-p^2}}{dp} = X$ functioni abſciſſæ x æquatione ad curvam inventus, erit $\int Tdx = X - y$ et ſumtis fluxionibus $Tdx = \dot{X}dx - dy$, qua $T = \dot{X} - \frac{dy}{dx}$ ubi tam $\dot{X}$ quam $\frac{dy}{dx}$ ſunt functiones abſciſſæ x. Si valor quantitatis $-\frac{dx\sqrt{1-p^2}}{dp} = P$ per p expreſſus, erit $\int Tdx = P - y$ ſumtiſque fluxionibus $Tdx = \dot{P}dp - dy$, qua $T = \frac{\dot{P}dp}{dx} - \frac{p}{\sqrt{1-p^2}}$ ubi $\frac{\dot{P}dp}{dx}$ functio eſt quantitatis p, nam $\frac{dp}{dx}$ per p exprimi poteſt.

Schol. 2. Hoc adhibito theoremate inveniri etiam poſſunt curvæ, ex data relatione inter T et x, F et x, F et y, F et z, et F et p. Poſita enim T functione quantitatis x, patet per curvarum quadraturas, aut perfectam aut imperfectam quantitatis Tdx obtineri integrationem. Sit $\int Tdx = \dot{X} + \int \dot{X}dx$ functioni vel algebraicæ vel ex parte tranſcendenti ipſius x, cujus terminis homogeneus valor ipſius $y = \int \ddot{X}dx$ capiatur, iſque ejus indolis ut $\int \overline{\dot{X} + \ddot{X}}dx$, vel quod idem eſt $y + \int Tdx =$

 X+

$X+\int \overline{X' + X'' dx}$ integratione absoluta habeatur, permanente $Tdz = Tdx\sqrt{1-X''^2}$ perfecte integrabili. Per theorema deinde habetur $\frac{dx}{X+\int X' + X'' dx} \left(= \frac{dx}{y+\int Tdx}\right) = -\frac{dp}{\sqrt{1-p^2}}$, et per integrationem $\int \frac{dx}{X+\int X' + X'' dx} + C = -\int \frac{dp}{\sqrt{1-p^2}}$, si ponatur $\int \frac{dx}{X+\int X' + X'' dx} + C = k$ et N basi logarithmorum hyperbolicorum, erit $\sqrt{1-p^2} = \frac{N^{k\sqrt{-1}} - N^{-k\sqrt{-1}}}{2\sqrt{-1}}$ et $p = \frac{N^{k\sqrt{-1}} + N^{-k\sqrt{-1}}}{2}$, $\sqrt{1-p^2}$ et p igitur sunt functiones ipsius x, quæ si ponantur $\sqrt{1-X'''^2}$ et X''', habetur $y \left(= \int \frac{pdx}{\sqrt{1-p^2}}\right) = \int \frac{X''' dx}{\sqrt{1-X'''^2}}$, æquatio qua curvæ internoscuntur.

Si sit $F = Y$ functioni quantitatis y erit per Cor. 2. $\frac{dy}{Y} \left(= \frac{dy}{F}\right) = -\frac{dp}{1-p^2}$ et integratione $\int \frac{dy}{Y} + \log. C = \log. \sqrt{1-p^2}$, ponatur $\int \frac{dy}{Y} = k$ et N logarithmorum basi, erit facto ad quantitates absolutas transitu $CN^k = \sqrt{1-p^2}$, $p = \sqrt{1-C^2N^{2k}}$ et $x \left(= \int \frac{dy\sqrt{1-p^2}}{p}\right) = \int \frac{CN^k dy}{\sqrt{1-C^2N^{2k}}}$, æquatio quæ indolem curvæ indigitat.

Si $F = Z$ functioni ipsius z erit $\frac{dz}{Z} \left(= \frac{dz}{F}\right) = -\frac{dp}{1-p^2}$ et integratione $\int \frac{dz}{Z} + \log. C = \log. \sqrt{\frac{1-p}{1+p}}$, et si $\int \frac{dz}{Z} = k$ et N basi logarithmica habetur $p = \frac{1-C^2N^{2k}}{1+C^2N^{2k}}$ et $y = \int pdz) = \int \frac{1-C^2N^{2k} dz}{1+C^2N^{2k}}$ qua curvæ cognoscuntur.

 Constat

Conſtat hinc quod quoties $X+\int \overset{I}{X}dx$ perfecta integratione habeatur $\int \frac{dx}{X+\int \overline{\overset{I}{X}+\overset{II}{X}dx}}$ per arcus circulares dum $\frac{\overset{III}{X}dx}{\sqrt{1-\overset{III}{X}^2}}$ abſolutam admittat integrationem curva ſit algebraica, ſi vero aliter evenerit tranſcendens.

Quoties $\frac{dy}{Y}$ ſit integrale logarithmicum et $\frac{CN^k dy}{\sqrt{1-C^2N^{2k}}}$ abſolutam admittat integrationem curva eſt algebraica, in aliis caſibus tranſcendens.

Et quoties $\int \frac{dz}{Z}$ per logarithmos inveniatur, $\frac{1-C^2N^{2k}dz}{1+C^2N^{2k}}$ abſolute ſit integrabilis pariter ac $\frac{2CN^k dz}{1+C^2N^{2k}}$ curva eſt algebraica, alias tranſcendens.

Exempl. 1. Si ſit variatio curvaturæ $T=\frac{3\cdot\overline{b^2-a^2}x\sqrt{a^2-x^2}}{a^3b}$ erit $\int Tdx \left(=\frac{\overline{a^2-b^2}\cdot\overline{a^2-x^2}\sqrt{a^2-x^2}}{a^3b}\right)=\frac{a\sqrt{a^2-x^2}}{b}-\frac{x^2\sqrt{a^2-x^2}}{ab}-\frac{b\sqrt{a^2-x^2}}{a}+\frac{bx^2\sqrt{a^2-x^2}}{a^3}$, ſi ponatur $y=\frac{b\sqrt{a^2-x^2}}{a}$ habetur $y+\int Tdx=\frac{a^4+\overline{b^2-a^2}x^2\sqrt{a^2-x^2}}{a^3b}$, adhibendo theorema $\frac{a^3bdx}{a^4+\overline{b^2-a^2}x^2\sqrt{a^2-x^2}}$ $\left(=\frac{dx}{y+\int Tdx}\right)=-\frac{dp}{\sqrt{1-p^2}}$ et integrando $\int\frac{a^3bdx}{a^4+\overline{b^2-a^2}x^2\sqrt{a^2-x^2}}+C=-\int\frac{dp}{\sqrt{1-p^2}}$, cujus termini ſunt arcus circulares quorum ſinus $\sqrt{1-p^2}=\frac{a\sqrt{a^2-x^2}}{\sqrt{a^4+\overline{b^2-a^2}x^2}}$ et coſinus $p=\frac{bx}{\sqrt{a^4+\overline{b^2-a^2}x^2}}$ evaneſcente arcu conſtanti C, quare $y\left(=\int\frac{pdx}{\sqrt{1-p^2}}\right)=\int\frac{bdx}{a\sqrt{a^2-x^2}}\Big)=\frac{b\sqrt{a^2-x^2}}{a}$ et in hoc caſu curva eſt ellipſis.

Exempl. 2. Sit jam variatio curvaturæ $T=\frac{2\sqrt{2ax+x^2}}{a}$ erit $\int Tdx=\frac{x\sqrt{2ax+x^2}}{a}+\int\frac{xdx}{\sqrt{2ax+x^2}}$ et posita $y=\int\frac{adx}{\sqrt{2ax+x^2}}$ perfecta integratione habetur $y+\int Tdx=\frac{\overline{a+x}\sqrt{2ax+x^2}}{a}$. Theorematis itaque auxilio erit $\frac{adx}{\overline{a+x}\sqrt{2ax+x^2}}\left(=\frac{dx}{y+\int Tdx}\right)=-\frac{dp}{\sqrt{1-p^2}}$, et integratione $\int\frac{adx}{\overline{a+x}\sqrt{2ax+x^2}}=C=-\int\frac{dp}{\sqrt{1-p^2}}$, si vero arcus ille constans $C=0$ cæteri sunt æquales eorumque sinus et cosinus, unde $\sqrt{1-p^2}=\frac{\sqrt{2ax+x^2}}{a+x}$, $p=\frac{a}{a+x}$ et $y\left(=\int\frac{pdx}{\sqrt{1-p^2}}\right)=\int\frac{adx}{\sqrt{2ax+x^2}}$, æquatio indicans curvam esse catenariam.

THEOREMA III.

Dicatur cosinus anguli BCD q, posito radio 1, cæterisque manentibus denominationibus erit $\frac{dy}{\int Tdy-x}=\frac{dq}{\sqrt{1-q^2}}$.

Est enim $\frac{ay}{R}=aq$, qua per $\sqrt{1-q^2}$ divisa, dat $\frac{dy}{R\sqrt{1-q^2}}=\frac{dq}{\sqrt{1-q^2}}$; et ob $1:\sqrt{1-q^2}::CD\,(R):CG=R\sqrt{1-q^2}$, sed $dz:dy::Tdz:Tdy$ cujus integrale est $AE=\int Tdy$, unde CG ($=AE-AB$) $=\int Tdy-x$, qua pro $R\sqrt{1-q^2}$ substituta, prodit $\frac{dy}{\int Tdy-x}=\frac{dq}{\sqrt{1-q^2}}$.

Cor. 1. Semper $Tdy-dx$ admittit perfectam integrationem. Etenim $Tdy=\frac{ddy\sqrt{1-q^2}}{dq}$ et $dx=\frac{qdy}{\sqrt{1-q^2}}$, quibus $Tdy-dx=\frac{ddy\sqrt{1-q^2}}{dq}-\frac{qdy}{\sqrt{1-q^2}}$ et integratione $\int Tdy-x=\frac{dy\sqrt{1-q^2}}{aq}$.

 Cor.

Cor. 2. Dicatur semichorda curvaturæ CG G, habetur $\frac{dy}{G}=\frac{dq}{\sqrt{1-q^2}}$, $\frac{dx}{G}=\frac{qdq}{1-q^2}$ et $\frac{dz}{G}=\frac{dq}{1-q^2}$.

Cor. 3. Dicatur cotangens anguli BCD t, et cosecans v erit $\frac{dy}{\int Tdy-x}=\frac{dt}{1+t^2}$ et $\frac{dy}{\int Tdy-x}=\frac{dv}{v\sqrt{v^2-1}}$.

Schol. 1. Quoniam $\int Tdy-x=\frac{dy\sqrt{1-q^2}}{dq}$ datur ex data relatione inter y et q, fit $\frac{dy\sqrt{1-q^2}}{dq}=Y$ functioni ordinatæ y erit $\int Tdy=Y-x$ sumtisque fluxionibus $Tdy=\dot{Y}dy-dx$ qua $T=\dot{Y}-\frac{dx}{dy}$ functioni ipsius y. Si autem $\frac{dy\sqrt{1-q^2}}{dq}=Q$ functioni ipsius q erit $\int Tdy=Q-x$ et sumtis fluxionibus $Tdy=\dot{Q}dq-dx$, qua habetur $T=\frac{\dot{Q}dq}{dy}-\frac{q}{\sqrt{1-q^2}}$ per q.

Schol. 2. Hujus theorematis auxilio elicere licet curvas data relatione inter T et y, G et y, G et x, G et z, et G et q. Si enim sit T functio ipsius y generaliter $\int Tdy=Y+\int\dot{Y}dy+A$, quæ functio est algebraica ipsius y quoties $\int\dot{Y}dy$ absolute sumi possit. Assumatur $x=\int\ddot{Y}dy$, tali ipsius y functioni ut non solum $\int Tdy-x=Y+\int\overline{\dot{Y}+\ddot{Y}}dy$ sed etiam $\int Tdz=\int Tdy\sqrt{1-\ddot{Y}^2}$ absoluta integratione habeantur, provenit vi theorematis $\frac{dy}{Y+\int\overline{\dot{Y}+\ddot{Y}}dy+A}\left(=\frac{dy}{\int Tdy-x}\right)=\frac{dq}{\sqrt{1-q^2}}$ et integratione $\int\frac{dy}{Y+\int\overline{\dot{Y}+\ddot{Y}}dy+A}+C=\int\frac{dq}{\sqrt{1-q^2}}$. Posita $\frac{dy}{Y+\int\overline{\dot{Y}+\ddot{Y}}dy+A}+C=l$ et N basi logarithmica erit $q=\frac{N^{l\sqrt{-1}}-N^{-l\sqrt{-1}}}{2\sqrt{-1}}$ et $\sqrt{1-q^2}$

$= \frac{N^{l\sqrt{-1}} + N^{-l\sqrt{-1}}}{2}$ quæ functiones sunt quantitatis y, quibus positis $\overset{|||}{Y}$ et $\sqrt{1 - \overset{|||}{Y}^2}$ prodit $x \left(= \int \frac{q\,dy}{\sqrt{1-q^2}}\right) = \int \frac{\overset{|||}{Y} dy}{\sqrt{1-\overset{|||}{Y}^2}}$ æquatio quæ indolem curvarum indicat.

Si $G = X$ functioni ipsius x erit per Cor. 2. $\frac{dx}{X} \left(= \frac{dx}{G}\right) = \frac{q\,dq}{1-q^2}$, et integratione $\log. CN^l \left(= \int \frac{dx}{X} + \log. C\right) = \log. \frac{1}{\sqrt{1-q^2}}$ si $\int \frac{dx}{X} = l$, exinde $\sqrt{1-q^2} = \frac{1}{CN^l}$, $q = \frac{\sqrt{C^2N^{2l}-1}}{CN^l}$ et $y \left(= \int \frac{dx\sqrt{1-q^2}}{q}\right) = \int \frac{dx}{\sqrt{C^2N^{2l}-1}}$, quæ curvæ naturam indigitat.

Si $G = Z$ functioni ipsius z erit $\frac{dz}{Z} \left(= \frac{dz}{G}\right) = \frac{dq}{1-q^2}$, et integratione $\log. CN^l \left(= \frac{dz}{Z} + C\right) = \log. \sqrt{\frac{1+q}{1-q}}$ si $\int \frac{dz}{Z} = l$, unde $q = \frac{C^2N^{2l}}{1+C^2N^{2l}}$ $\sqrt{1-q^2} = \frac{2CN^l}{1+C^2N^{2l}}$ $x \left(= \int q\,dz\right) = \int \frac{C^2N^{2l}-1\,dz}{1+C^2N^{2l}}$ et $y \left(= \int dz\sqrt{-1q^2}\right) = \int \frac{2CN^l dz}{1+C^2N^{2l}}$ quibus curvæ cognoscuntur.

Patet hinc quod quando $Y + \int \overset{|}{Y} dy$ algebraice habeatur $\int \frac{dy}{Y + \int \overset{|}{Y} + \overset{||}{Y} dy + A}$ per quadraturam circuli, et $\int \frac{\overset{|||}{Y} dy}{\sqrt{1-\overset{|||}{Y}^2}}$ etiam obtineatur algebraice, curvæ evadunt algebraicæ, secus vero transcendentes.

Quando $\int \frac{dx}{X}$ vel $\int \frac{dz}{Z}$ obtineatur per logarithmos, et $\int \frac{dx}{\sqrt{C^2N^{2l}-1}}$, vel tam $\int \frac{C^2N^{2l}-1\,dz}{1+C^2N^{2l}}$ quam $\int \frac{2CN^l dz}{1+C^2N^{2l}}$ absoluta integratione, curvæ erunt algebraicæ.

Exempl.

Exempl. 1. Sit index variationis curvaturæ $T = \frac{6y}{a}$ erit $\int T dy = \frac{3y^2}{a} + A$, si quantitas illa constans $A = \frac{a}{2}$ quod evenit quum $\int T dy = \frac{a}{2}$ et $y = 0$; sumatur $x = \frac{y^2}{a}$ erit vi theorematis $\frac{2ady}{a^2+4y^2} \left(= \frac{dy}{\int T dy - x}\right) = \frac{dq}{\sqrt{1-q^2}}$ et integratione $\int \frac{2ady}{a^2+4y^2} + C = \int \frac{dq}{\sqrt{1-q^2}}$, cujus æquationis termini quoniam sint arcus circulares quorum sinus $q = \frac{2y}{\sqrt{a^2+4y^2}}$ et cosinus $\sqrt{1-q^2} = \frac{a}{\sqrt{a^2+4y^2}}$, arcu constanti $C = 0$, obtinetur $x \left(= \int \frac{qdy}{\sqrt{1-q^2}}\right) = \frac{y^2}{a}$ æquatio pro parabola Apolloniana.

Exempl. 2. Si sit $T = \frac{a^2}{y\sqrt{a^2-y^2}}$ habetur $\int T dy = \int \frac{dy\sqrt{a^2-y^2}}{y} - \sqrt{a^2-y^2} + A$, si quantitas illa constans $A = 0$ quod evenit quum $\int T dy = 0$ et $y = a$, et assumatur $x = \int \frac{dy\sqrt{a^2-y^2}}{y}$, evadit per theorema $-\frac{dy}{\sqrt{a^2-y^2}} \left(= \frac{dy}{\int T dy - x}\right) = \frac{dq}{\sqrt{1-q^2}}$, et per integrationem $-\int \frac{dy}{\sqrt{a^2-y^2}} + C = \int \frac{dq}{\sqrt{1-q^2}}$, quorum arcuum sinus $q = \frac{\sqrt{a^2-y^2}}{a}$ et cosinus $\sqrt{1-q^2} = \frac{y}{a}$ si constans ille $C = 0$, atque inde $dx \left(\frac{qdy}{\sqrt{1-q^2}}\right) = \frac{dy\sqrt{1-y^2}}{y}$ qua patet curvam esse tractoriam.

THEOREMA IV.

Dicatur summa tangentium angulorum HCD et BCD H, et differentia tangentium angulorum HCD et CKB K, retentis reliquis denominationibus erit $\frac{dx}{\int H dx} = -\frac{dp}{\sqrt{1-p^2}}$ et $\frac{dy}{\int K dy} = \frac{dq}{\sqrt{1-q^2}}$.

Quoniam

Quoniam $dy = \frac{pdx}{\sqrt{1-p^2}}$ erit $dy + Tdx = \overline{T + \frac{p}{\sqrt{1-p^2}}}\, dx$ et quum $H = T + \frac{p}{\sqrt{1-p^2}}$ habetur $dy + Tdx = Hdx$. Eodem modo quum $dx = \frac{qdy}{\sqrt{1-q^2}}$ erit $\int Tdy - dx = T - \frac{q}{\sqrt{1-q^2}}\, dy$, ſed $K = T - \frac{q}{\sqrt{1-q^2}}$, unde $\int Tdy - x = Kdy$. Per theorema igitur 2 et 3 provenit $\frac{dx}{\int Hdx} = -\frac{dp}{\sqrt{1-p^2}}$ et $\frac{dy}{\int Kdy} = \frac{dq}{\sqrt{1-q^2}}$.

Cor. Si ſit ut antea tangens anguli BCD r, cotangens t, ſecans s, et coſecans v, erit $\frac{dx}{\int Hdx} = -\frac{dr}{r+r^2}$ et $\frac{dx}{\int Hdx} = -\frac{ds}{s\sqrt{s^2-1}}$, $\frac{dy}{\int Kdy} = \frac{dt}{1+t^2}$ et $\frac{dy}{\int Kdy} = \frac{dv}{v\sqrt{v^2-1}}$.

Schol. Ope hujus theorematis invenire licet curvas, data relatione inter H et x atque K et y. Itaque ſit $H = X$ functioni ipſius x erit $\int Hdx = \int Xdx + A$, vi theorematis $\frac{dx}{\int Xdx + A}$ $\left(= \frac{dx}{\int Hdx}\right) = -\frac{dp}{\sqrt{1-p^2}}$, et integratione $\int \frac{dx}{\int Xdx + A} + C = -\int \frac{dp}{\sqrt{1-p^2}}$. Poſita $\int \frac{dx}{\int Xdx + A} + C = m$, et N logarithmorum baſi prodit $\sqrt{1-p^2} = \frac{N^{m\sqrt{-1}} - N^{-m\sqrt{-1}}}{2\sqrt{-1}}$ et $p = \frac{N^{m\sqrt{-1}} + N^{-m\sqrt{-1}}}{2}$, quibus functionibus quantitatis x poſitis $\sqrt{1-\overset{\prime}{X}^2}$ et $\overset{\prime}{X}$ provenit æquatio $y\ \left(= \int \frac{pdx}{\sqrt{1-p^2}}\right) = \frac{\overset{\prime}{X}dx}{\sqrt{1-\overset{\prime}{X}^2}}$ naturam curvarum exprimens.

Si $K = Y$ functioni quantitatis y, eadem calculandi ratione habetur $x\ \left(= \int \frac{qdy}{\sqrt{1-q^2}}\right) = \frac{\overset{\prime}{Y}dy}{\sqrt{1-\overset{\prime}{Y}^2}}$ æquatio qua curvæ cognoſcuntur.

Quando

Quando $\int Xdx$ vel $\int Ydy$ absoluta integratione, $\int \frac{dx}{\int Xdx + A}$ vel $\int \frac{dy}{\int Ydy + A}$ per rectificationem circuli, et $\int \frac{\dot{X}dx}{\sqrt{1-\dot{X}^2}}$ vel $\int \frac{\dot{Y}dy}{\sqrt{1-\dot{Y}^2}}$ integratione perfecta obtineantur, curva est algebraica.

Exempl. 1. Si sit $H = \frac{a+12x}{2\sqrt{a}\sqrt{x}}$ erit $\int Hdx = \frac{\overline{a+4x\sqrt{x}}}{\sqrt{a}} + A$, et posita $A = 0$ habetur per theorema $\frac{dx\sqrt{a}}{a+4x\sqrt{x}} \left(= \frac{dx}{\int Hdx}\right) = -\frac{dp}{\sqrt{1-p^2}}$ et per integrationem $\int \frac{dx\sqrt{a}}{a+4x\sqrt{x}} + C = \int \frac{dp}{\sqrt{1-p^2}}$, cujus termini quum sint arcus circulares quorum sinus $\sqrt{1-p^2} = \frac{2\sqrt{x}}{\sqrt{a+4x}}$ et cosinus $p = \frac{\sqrt{a}}{\sqrt{a+4x}}$, posita $C = 0$, obtinetur $y \left(= \int \frac{pdx}{\sqrt{1-p^2}}\right) = \sqrt{ax}$, quæ parabolam Apolloniam exprimit.

Exempl. 2. Sit $H = \frac{2a^4 - x^4}{ax^2\sqrt{a^2-x^2}}$ erit $\int Hdx = \frac{\overline{x^2 - 2a^2\sqrt{a^2-x^2}}}{ax} + A$, et si $A = 0$, per theorema $\frac{axdx}{x^2 + 2a^2\sqrt{a^2-x^2}} \left(= \frac{dx}{\int Hdx}\right) = -\frac{dp}{\sqrt{1-p^2}}$ et per integrationem $\int \frac{axdx}{x^2 - 2a^2\sqrt{a^2-x^2}} + C = -\int \frac{dp}{\sqrt{1-p^2}}$, et si $C = 0$, $\sqrt{1-p^2} = \frac{\sqrt{a^2-x^2}}{\sqrt{2a^2-x^2}}$ $p = \frac{a}{\sqrt{2a^2-x^2}}$ et $y \left(= \int \frac{pdx}{\sqrt{1-p^2}}\right) = \int \frac{adx}{\sqrt{a^2-x^2}}$ æquatio pro curva sinuum.

Exempl. 3. Si sit $K = \frac{\overline{5a^2 + 6y^2}.y}{a\sqrt{a^2+y^2}}$ erit $\int K dy = \frac{\overline{a^2 + 2y^2\sqrt{a^2+y^2}}}{a^2} + A$, si $A = 0$ habetur per theorema $\frac{a^2dy}{a^2 + 2y^2\sqrt{a^2+y^2}} \left(= \frac{dy}{\int Kdy}\right) = \frac{dq}{\sqrt{1-q^2}}$ et integratione $\int \frac{a^2dy}{a^2 + 2y^2\sqrt{a^2+y^2}} + C = -\int \frac{dp}{\sqrt{1-p^2}}$, qua $q = \frac{y}{\sqrt{a^2+2y^2}}$, $\sqrt{1-q^2} = \frac{\sqrt{a^2+y^2}}{\sqrt{a^2+2y^2}}$, si $C = 0$, unde $x \left(= \int \frac{qdy}{\sqrt{1-q^2}}\right) = \sqrt{a^2+y^2}$ æquatio pro hyperbola æquilatera.

Exempl.

Exempl. 4. Sit $K = \frac{y}{\sqrt{a^2 - y^2}}$ erit $\int K dy = A - \sqrt{a^2 - y^2}$ et si $A = 0$, per theorema $-\frac{dy}{\sqrt{a^2 - y^2}} \left(= \frac{dy}{\int K dy} \right) = \frac{dq}{\sqrt{1 - q^2}}$ et per integrationem $-\int \frac{dy}{\sqrt{a^2 - y^2}} + C = \int \frac{dq}{\sqrt{1 - q^2}}$ qua $q = \frac{\sqrt{a^2 - y^2}}{a}$, $\sqrt{1 - q^2} = \frac{y}{a}$ et $dx \left(= \frac{q dy}{\sqrt{1 - q^2}} \right) = \frac{dy \sqrt{a^2 - y^2}}{y}$ quæ Tractoriam exprimit.

THEOREMA V.

Designetur productum tangentium angulorum HCD et BCD per U, et angulorum HCD et CKB per V cæteris manentibus erit $\frac{dx}{\int U dx - x} = -\frac{dp}{p}$ et $\frac{dy}{Y + \int V dy} = \frac{dq}{q}$.

Quoniam $dy = \frac{p dx}{\sqrt{1 - p^2}}$ et $U = \frac{Tp}{\sqrt{1 - p^2}}$ erit $T dy \left(= \frac{T p dx}{\sqrt{1 - p^2}} \right) = U dx$, et integratione $\int T dy = \int U dx$ qua $\int T dy - x = \int U dx - x$. Et quoniam $dx = \frac{q dy}{\sqrt{1 - q^2}}$ et $V = \frac{Tq}{\sqrt{1 - q^2}}$ erit $T dx \left(= \frac{T q dy}{\sqrt{1 - q^2}} \right) = V dy$, $\int T dx = \int V dy$ et $y + \int T dx = y + \int V dy$. Theoremate 2. et 3. prodit $\frac{dx}{\int U dx - x} = -\frac{dp}{p}$ et $\frac{dy}{y + \int V dy} = \frac{dq}{q}$.

Cor. Si anguli BCD tangens, cotangens, &c. designentur ut antea, habetur $\frac{dx}{\int U dx - x} = -\frac{dr}{r \cdot 1 + r^2}$, $\frac{dy}{y + \int V dy} = -\frac{dt}{t \cdot 1 + t^2}$, &c.

Schol. Per hoc theorema curvæ inveniuntur ex data relatione inter U et x, atque inter V et y. Si enim sit $U = X$ functioni ipsius x erit $\int U dx = \int X dx + A$, per theorema $\frac{dx}{\int X dx - x + A} \left(= \frac{dx}{\int U dx - x} \right) = -\frac{dp}{p}$, et per integrationem $\int \frac{dx}{\int X dx - x + A} + \log. C =$

log. $\frac{1}{p}$. Ponatur $\int \frac{dx}{\int Xdx - x + A} = n$ et N basi logarithmica, erit $\frac{1}{p} = CN^n$, $p = \frac{1}{CN^n}$, $\sqrt{1-p^2} = \frac{\sqrt{C^2N^{2n}-1}}{CN^n}$ et $y\ (= \frac{pdx}{\sqrt{1-p^2}}) = \int \frac{dx}{\sqrt{C^2N^{2n}-1}}$ qua æquatione curvarum indoles innotescit.

Si V = Y functioni ipsius y, eadem calculandi ratione provenit $x\ (= \int \frac{qdy}{\sqrt{1-q^2}}) = \int \frac{CN^n dy}{\sqrt{1-C^2N^{2n}}}$ qua curvæ cognoscuntur.

Evidens hinc est quod quoties $\int Xdx$ vel $\int Ydy$ algebraice $\int \frac{dx}{\int Xdx - x + A}$ vel $\int \frac{dy}{y + \int Ydy + A}$ per logarithmos, atque $\int \frac{dx}{\sqrt{C^2N^{2n}-1}}$ vel $\int \frac{CN^n dy}{\sqrt{1-C^2N^{2n}}}$ integratione absoluta, obtineantur, curva est algebraica.

Exempl. 1. Si sit U = 3 erit $\int Udx = 3x + A$, si vero $\int Udx = \frac{a}{2}$ quando $x = 0$ erit $A = \frac{a}{2}$ et $\int Udx - x = \frac{a+4x}{2}$. Per theorema igitur $\frac{2dx}{a+4x}\ (= \frac{dx}{\int Udx - x}) = -\frac{dp}{p}$ et per integrationem log. $\sqrt{a+4x}$ + log. C = log. $\frac{1}{p}$, posita $p = 1$ dum $x = 0$ log. C = − log. $\sqrt{a}$, unde facto a logarithmis transitu $\frac{\sqrt{a+4x}}{\sqrt{a}} = \frac{1}{p}$, qua $p = \frac{\sqrt{a}}{\sqrt{a+4x}}$, $\sqrt{1-p^2} = \frac{2\sqrt{x}}{\sqrt{a+4x}}$ et $y\ (= \int \frac{pdx}{\sqrt{1-p^2}}) = \int \frac{dx\sqrt{a}}{2\sqrt{x}}) = \sqrt{ax}$ æquatio pro Parabola Apolloniana.

Exempl. 2. Sit $U = \frac{x^3 - 4a^3}{x\sqrt{x}}$ erit $\int Udx = \frac{x^3 - 2a^3}{3x^2} + A$, si autem $\int Udx = 0$ et $x = a\sqrt[3]{2}$, erit $A = 0$ et $\int Udx - x = \frac{2 \cdot a^3 - x^3}{3x^2}$. Vi igitur theorematis erit $\frac{3x^2dx}{2a^3 - x^3}\ (= \frac{dx}{\int Udx - x}) = -\frac{dp}{p}$, et integratione

 log.

log. $\frac{a\sqrt{a}}{\sqrt{a^3-x^3}}$ + log. C = log. $\frac{1}{p}$ qua $p = \frac{\sqrt{a^3-x^3}}{a\sqrt{a}}$; $\sqrt{1-p^2} = \frac{x\sqrt{x}}{a\sqrt{a}}$ et $y \left(= \int \frac{pdx}{\sqrt{1-p^2}}\right) = \frac{dx\sqrt{a^3-x^3}}{x\sqrt{x}}$ æquatio ad curvam quæſitam.

Exempl. 3. Si $V = -\frac{1}{2}$ erit $\int Vdy = A - \frac{y}{2}$, poſita $\int Vdy = 0$ et $y = 0$ erit $A = 0$ et $y + \int Vdy = \frac{y}{2}$. Per theorema obtinetur $\frac{2dy}{y} \left(= \frac{dy}{y + \int Vdy}\right) = \frac{dq}{q}$ et per integrationem log. y^2 + log. C = log. q, ſi $q = 1$ et $y = a$ erit log. C = −log. a^2, unde $q = \frac{y^2}{a^2}$, $\sqrt{1-q^2} = \frac{\sqrt{a^4-y^4}}{a^2}$ atque $dx \left(= \frac{qdy}{\sqrt{1-q^2}}\right) = \frac{y^2dy}{\sqrt{a^4-y^4}}$, curva ergo eſt Elaſtica,

Exempl. 4. Sit $V = \frac{a^2-2y^2}{y^2}$ erit $\int Vdy = A - \frac{a^2+2y^2}{y}$, ſi $\int Vdy = -3a$ et $y = a$ erit $A = 0$, indeque $y + \int Vdy = -\frac{a^2+y^2}{y}$. Theorematis ope habetur $-\frac{ydy}{a^2+y^2} \left(= \frac{dy}{y+\int Vdy}\right) = \frac{dq}{q}$ et integratione log. $\frac{1}{\sqrt{a^2+y^2}}$ + log. C = log. q, ſi $q = 1$ et $y = 0$ erit log. C = log. a et exinde $q = \frac{a}{\sqrt{a^2+y^2}}$, $\sqrt{1-q^2} = \frac{y}{\sqrt{a^2+y^2}}$ et $dx \left(= \frac{qdy}{\sqrt{1-q^2}}\right) = \frac{ady}{y}$ æquatio pro Logarithmica.

THEOREMA VI.

Dicatur ED L, et AE M, retentis præterea adhibitis denominationibus erit $\frac{dL}{T} = dx$ et $\frac{dM}{T} = dy$.

Quoniam $dz : dx :: Tdz$ (dR) : Tdx habetur $dL = Tdx$ et

$\frac{dL}{T} = dx$. Et quoniam $dz : dy :: Tdz\ (dR) : Tdy$ obtinetur $dM = Tdy$ et $\frac{dM}{T} = dy$.

Cor. Quum $Tdy = Udx$ et $Tdx = Vdy$, erit ſubſtitutione $\frac{dM}{U} = dx$ et $\frac{dL}{V} = dy$.

Schol. Hoc adhibito theoremate inveniri poſſunt curvæ data relatione inter T et L, T et M, atque inter U et M et V et L. Ponatur $L = \overset{1}{T}$ functioni quantitatis T habetur per theorema $\frac{d\overset{1}{T}}{T}\ (= \frac{dL}{T}) = dx$ et integratione $\int \frac{d\overset{1}{T}}{T} + C = x$ qua T per x datur. Curvæ deinde per theorema 2. elici poſſunt.

Si $M = \overset{1}{T}$ ipſius T functioni, habetur eodem modo T per y. Si $M = \overset{1}{U}$ functioni ipſius U, obtinetur U per x, et ſi $L = \overset{1}{V}$ functioni quantitatis V, datur V per y. Per theorema deinde 3. et 5. curvæ inveniuntur.

Evidens quidem eſt quod curvæ eſſe non poſſunt algebraicæ niſi $\int \frac{dL}{T}$, $\int \frac{dM}{T}$, $\int \frac{dM}{U}$ vel $\int \frac{dL}{V}$, obtineantur integratione abſoluta.

Exempl. 1. Si ſit $L = \frac{aT^3}{54}$ erit $dL = \frac{aT^2dT}{18}$, et per hoc theorema $\frac{aTdT}{18}\ (= \frac{dL}{T}) = dx$ et integratione $\frac{aT^2}{36} + C = x$ qua $T = \frac{6\sqrt{x}}{\sqrt{a}}$, ſi $C = 0$. Per theorema 2. reperitur $y = \sqrt{ax}$, æquatio pro Parabola Apolloniana.

Exempl. 2. Si ſit $M = -\int \frac{aT^2dT}{2 \cdot \overline{1+T^2}|^2}$ erit $dM = -\frac{aT^2dT}{2 \cdot \overline{1+T^2}|^2}$ et ope theorematis $-\frac{aTdT}{2 \cdot \overline{1+T^2}|^2}\ (= \frac{dM}{T}) = dy$, et integratione $\frac{a}{4 \cdot \overline{1+T^2}}$ $+ C = y$, qua ſi $C = 0$, $T = \frac{\sqrt{a-4y}}{2\sqrt{y}}$. Per theorema 3. habetur

$dx =$

$dx = \frac{2dy\sqrt{y}}{\sqrt{a-4y}}$, æquatio pro Cycloide ordinaria.

Exempl. 3. Sit $L = -a\sqrt{V}$ erit $dL = -\frac{adV}{2\sqrt{V}}$ et per theorema $-\frac{adV}{2V\sqrt{V}}\ (=\frac{dL}{V}) = dy$ et integratione $\frac{a}{\sqrt{V}} + C = y$, et si $C = 0$, habetur $V = \frac{a^2}{y^2}$ et deinde per theorema 5. $dx = \frac{dy\sqrt{a^2-y^2}}{y}$, qua constat curvam esse Tractoriam.

THEOREMA VII.

Dicatur ut antea CF F et CG G, et summa tangentium angulorum HCD et BCD, H, et differentia tangentium angulorum HCD et CKB, K, erit $\frac{dF}{H} = dx$ et $\frac{dG}{K} = dy$.

Quoniam $dF\ (= dy + Tdx) = Hdx$ et $dG\ (= \int Tdy - x) = Kdy$ provenit $\frac{dF}{H} = dx$ et $\frac{dG}{K} = dy$.

Cor. Quum $F = -\frac{dx\sqrt{1-p^2}}{dp}$ et $G = \frac{dy\sqrt{1-q^2}}{dp}$ provenit divisione $\frac{dF}{FH} = -\frac{dp}{\sqrt{1-p^2}}$ atque $\frac{dG}{GK} = \frac{dq}{\sqrt{1-q^2}}$.

Schol. Auxilio hujus theorematis inveniuntur curvæ ex data relatione inter F et H, G et K, H et p atque K et q. Nam si sit $F = \acute{H}$ functioni ipsius H, vel $G = \acute{K}$ functioni ipsius K, habetur per theorema $\frac{d\acute{H}}{H}\ (=\frac{dF}{H}) = dx$ et integratione $\int \frac{d\acute{H}}{H} + C = x$ qua H per x datur. Eodem modo $\frac{d\acute{K}}{K}\ (=\frac{dG}{K}) = dy$ et integratione $\int \frac{d\acute{K}}{K} + C = y$ qua K per y obtinetur. Theorema 4. ulterius progredienti viam monstrat ad curvas inveniendas.

Patet

Patet quod curva non ſit algebraica niſi $\int\frac{dH}{H}$ vel $\int\frac{dK}{K}$ obtineantur perfecta integratione.

Exempl. 1. Si ſit $F=\frac{a}{\sqrt{1+H^2}}$ habetur per theorema – $\frac{adH}{\overline{1-H^2}|^{\frac{3}{2}}}\left(=\frac{dF}{H}\right)=dx$, et integratione $\frac{aH}{\sqrt{1-H^2}}+C=-x$ qua $H=-\frac{x}{\sqrt{a^2-x^2}}$, poſita $C=0$. Per theorema deinde 4. provenit $y=\sqrt{a^2-x^2}$ æquatio pro circulo.

Exempl. 2. Sit $F=\frac{a.\overline{H^3+H^2+6\sqrt{H^2-12}}}{108}$, erit per theorema $\frac{a.\overline{H^2-6+H\sqrt{H^2-12}}.dH}{36\sqrt{H^2-12}}\left(=\frac{dF}{H}\right)=dx$ et integratione facta $\frac{a.\overline{H^2-6+H\sqrt{H^2-12}}}{72}+C=x$, et poſita $C=0$ habetur $H=\frac{a+12x}{2\sqrt{a}\sqrt{x}}$, unde per theorema 4. prodit $y=\sqrt{ax}$ æquatio pro Parabola Apolloniana.

Exempl. 3. Sit $G=-\frac{a.\overline{4+K^2}}{4}$ erit per theorema $\frac{adK}{2}\left(=\frac{dG}{K}\right)=dy$, et integratione $\frac{aK}{2}+C=y$, et ſi $C=0$ $K=\frac{2y}{a}$ unde per theorema 4. $dx=\frac{ady}{y}$, qua conſtat curvam eſſe Logarithmicam.

THEOREMA VIII.

Dicatur ut antea productum tangentium angulorum HCD et BCD U, et productum tangentium angulorum HCD et CKB V manentibus reliquis denominationibus erit $\frac{dG}{U-1}=dx$ et $\frac{dF}{1+V}=dy$.

Quoniam

Quoniam $G = \int Tdy - x$ erit $dG = Tdy - dx$, ſed $Tdy = Udx$, unde $dG = \overline{U - 1}\,dx$ et $\frac{dG}{U-1} = dx$. Eeodem modo quum $F = y + \int Tdx$ erit $dF = dy + Tdx$, ſed $Tdx = Vdy$ quare $dF = \overline{1 + V}\,dy$ et $\frac{dF}{1+V} = dy$.

Cor. Quoniam $G = \frac{dy\sqrt{1-q^2}}{dq}$ et $F = -\frac{dx\sqrt{1-p^2}}{dp}$, habetur ſubſtitutione debita $\frac{dG}{G \cdot \overline{U-1}} = -\frac{dp}{p}$ et $\frac{dF}{F^2\,\overline{1+V}} = \frac{dq}{q}$.

Schol. Ope hujus theorematis indagantur curvæ data relatione inter G et U vel inter F et V. Nam ſi ſit $G = \acute{U}$ functioni quantitatis U vel $F = \acute{V}$ functioni quantitatis V obtinetur per theorema in caſu priori $\frac{d\acute{U}}{U-1}$ $\left(= \frac{dG}{U-1}\right) = dx$ et integratione $\int \frac{d\acute{U}}{U-1} + C = x$, qua U per x habetur; in poſteriori $\frac{d\acute{V}}{1+V}$ $\left(= \frac{dF}{1+V}\right) = dy$ et integratione $\int \frac{d\acute{V}}{1+V} + C = y$, qua V habetur per y. Per theorema deinde 5. curvæ cognoſcuntur.

Datur etiam per Cor. U in p, et V in q, et conſequenter T in p vel q, nam $U = \frac{Tp}{\sqrt{1-p^2}}$ et $V = \frac{Tq}{\sqrt{1-q^2}}$.

Conſtat hinc quod curvæ non ſint algebraicæ niſi $\int \frac{d\acute{U}}{U-1}$ vel $\int \frac{d\acute{V}}{1+V}$ obtineantur integratione abſoluta.

Exempl. 1. Si ſit $G = \frac{a \cdot \overline{U-3}}{2}$ erit per theorema $\frac{adU}{2\,\overline{U-1}}$ $\left(= \frac{dG}{U-1}\right) = dx$ et integratione log. $1 - U + \text{log.}\, C = \frac{2x}{a}$ et ſi $C = \frac{a^2}{2}$

log.

log. $\frac{a^2 . \overline{1-U}}{2} = \frac{2x}{a}$ et $\frac{a . \overline{1-U}}{2} = N^{\frac{2x}{a}}$ qua $U = \frac{a^2 - 2N^{\frac{2x}{a}}}{a^2}$. Per theorema deinde 5. habetur $dy = \frac{dxN^{\frac{x}{a}}}{a}$ qua conſtat curvam eſt Logarithmicam.

Exempl. 2. Si ſit $T = \frac{a . \overline{V-1}\sqrt{V+2}}{3\sqrt{3}}$ erit per theorema $\frac{adV}{2\sqrt{3}\sqrt{V+2}} \left(= \frac{dF}{1+V}\right) = dy$ et per integrationem $\frac{a\sqrt{V+2}}{\sqrt{3}} = y$ qua $V = \frac{3y^2 - 2a^2}{a^2}$; et per theorema 5. $dx = \frac{dy\sqrt{y^2 - a^2}}{a}$, æquatio ad curvam cujus conſtructio a quadratura hyperbolæ dependet.

THEOREMA IX.

Sint LC et lc duæ curvæ eandem habentes Evolutam QD, dicatur radiorum oſculi CD cD conſtans differentia cC b, curvæ lc variatio curvaturæ S, ceteriſque ut antea manentibus erit $\frac{dR}{R-bS} = -\frac{dp}{\sqrt{1-p^2}}$.

Quoniam radius curvaturæ DH evolutæ ſit $RT = R - bS$, erit $\frac{1}{R-bS} = \frac{1}{RT}$, quæ per $dR\ (= Tdz) = -\frac{RTdp}{\sqrt{1-p^2}}$ multiplicata, moſtrat eſſe $\frac{dR}{R-bS} = -\frac{dp}{\sqrt{1-p^2}}$.

Cor. Si ſint ut antea tangens anguli BCD r et ſecans s, habetur $\frac{dR}{R-bS} = -\frac{dr}{1+r^2}$ et $\frac{dR}{R-bS} = -\frac{ds}{S\sqrt{S^2-1}}$.

Schol. Subſidio hujus theorematis invenire licet curvas, data relatione inter S et R vel inter S et T nam $\frac{S}{T} = \frac{R}{R-b}$. Itaque ſi

ponatur

ponatur $S = \acute{R}$ functioni radii curvedinis R, erit $\frac{dR}{R - b\acute{R}} \left(= \frac{dR}{R - bS}\right) = -\frac{dp}{\sqrt{1-p^2}}$, et integratione $\int \frac{dR}{R - b\acute{R}} + C = -\int \frac{dp}{\sqrt{1-p^2}}$. Sit $\int \frac{dR}{R - b\acute{R}} + C = f$ et N logarithmorum basi habetur $\sqrt{1-p^2} = \frac{N^{f\sqrt{-1}} - N^{-f\sqrt{-1}}}{2\sqrt{-1}}$ et $p = \frac{N^{f\sqrt{-1}} + N^{-f\sqrt{-1}}}{2}$ functionibus quantitatis R, quibus R per p exprimi potest. Per theorema igitur 1. curvas internoscere valemus.

Si $R = \acute{S}$ functioni quantitatis S habetur $\frac{d\acute{S}}{\acute{S} - bS} \left(= \frac{dR}{R - bS}\right) = -\frac{dp}{\sqrt{1-p^2}}$, et integratione $\int \frac{d\acute{S}}{\acute{S} - bS} + C = -\int \frac{dp}{\sqrt{1-p^2}}$, posita $\int \frac{d\acute{S}}{\acute{S} - bS} + C = g$, erit $\sqrt{1-p^2} = \frac{N^{g\sqrt{-1}} - N^{-g\sqrt{-1}}}{2 - \sqrt{1}}$ et $p = \frac{N^{g\sqrt{-1}} + N^{-g\sqrt{-1}}}{2}$ quibus S per p datur. Per theoremata Partis I. invenire licet curvas omnes eandem evolutam habentes.

Hinc videtur, quod curvæ non sint algebraicæ nisi $\int \frac{dR}{R - b\acute{R}}$ vel $\int \frac{d\acute{S}}{\acute{S} - bS}$ per circuli rectificationem obtineatur.

Exempl. 1. Si sit $S = \frac{2R}{\sqrt{a}\,.\,\sqrt{R-a}}$ supposita $b = a$, erit per theorema $\frac{dR\sqrt{a}}{2R\sqrt{R-a}} \left(= \frac{dR}{R - bS}\right) = -\frac{dp}{\sqrt{1-p^2}}$ et integratione $\int \frac{dR\sqrt{a}}{2R\sqrt{R-a}} + C = -\int \frac{dp}{\sqrt{1-p^2}}$, si vero arcus ille constans $C = 0$ erit $\sqrt{1-p^2} = \frac{\sqrt{R-a}}{\sqrt{R}}$ qua $R = ap^2$, et per Cor. 1. Theor. 1. habetur $dy = \frac{adx}{\sqrt{x^2 - a^2}}$, æquatio pro Catenaria.

Exempl. 2. Sit $S=\frac{5a^2+R^2}{a\cdot\overline{a-5R}}$, posita $b=\frac{a}{5}$ erit per theorema $-\frac{adR}{a^2+R^2}\left(=\frac{dR}{R-bS}\right)=-\frac{dp}{\sqrt{1-p^2}}$ et facta integratione $-\int\frac{adR}{a^2+R^2}+C=-\int\frac{dp}{\sqrt{1-p^2}}$, qua si $C=0$, habetur $\sqrt{1-p^2}=\frac{R}{\sqrt{a^2+R^2}}$ et $R=\frac{a\sqrt{1-p^2}}{p}$. Per theorema 1. $dx=\frac{dy\sqrt{a^2-y^2}}{y}$ qua constat curvam esse Tractoriam.

Fig.1.

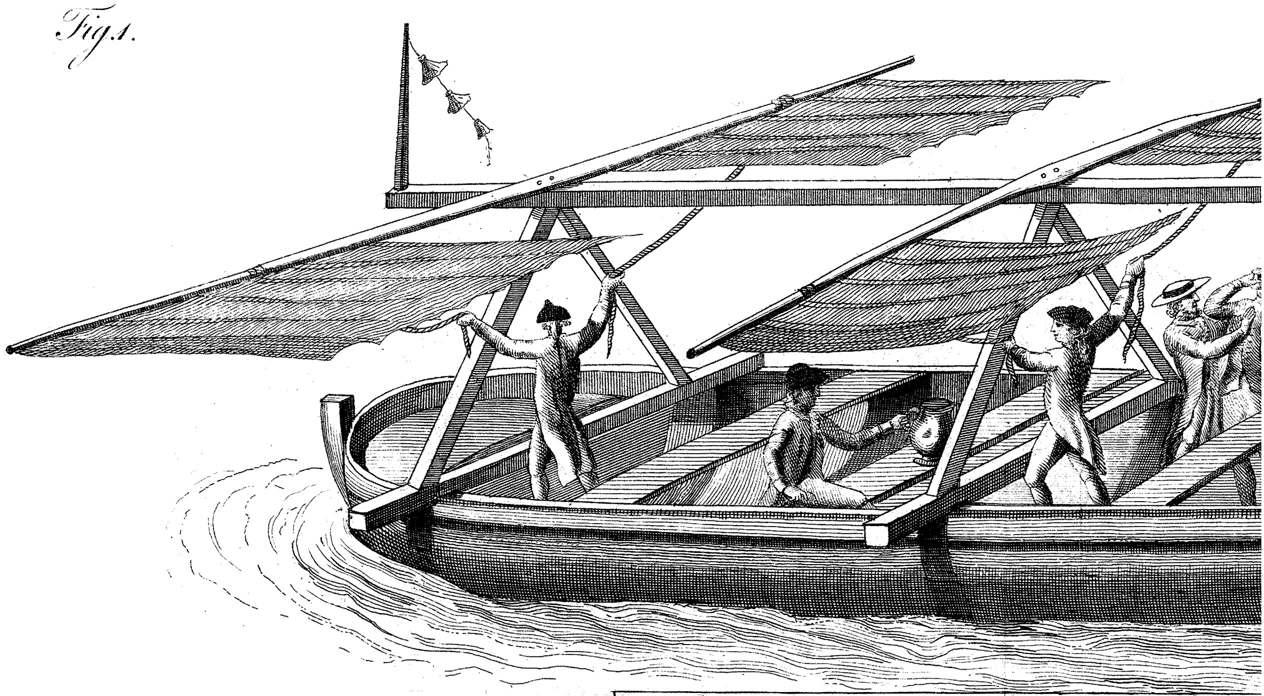
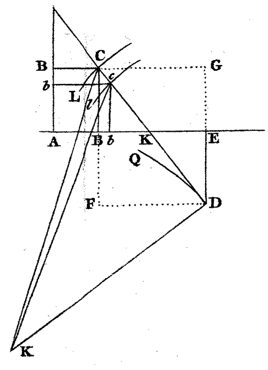
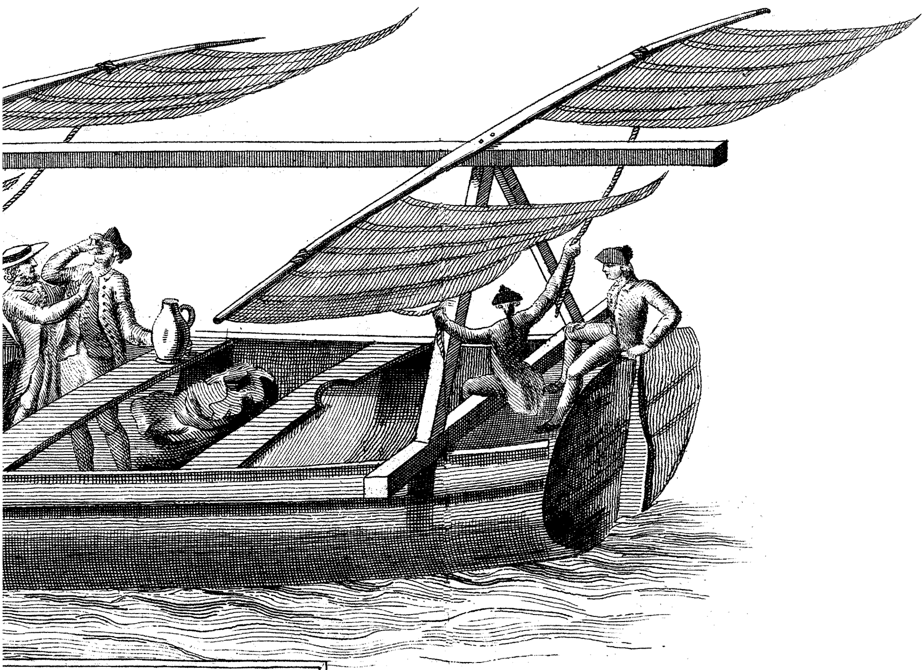


Fig.2.





Basire s.

Fig. 1.

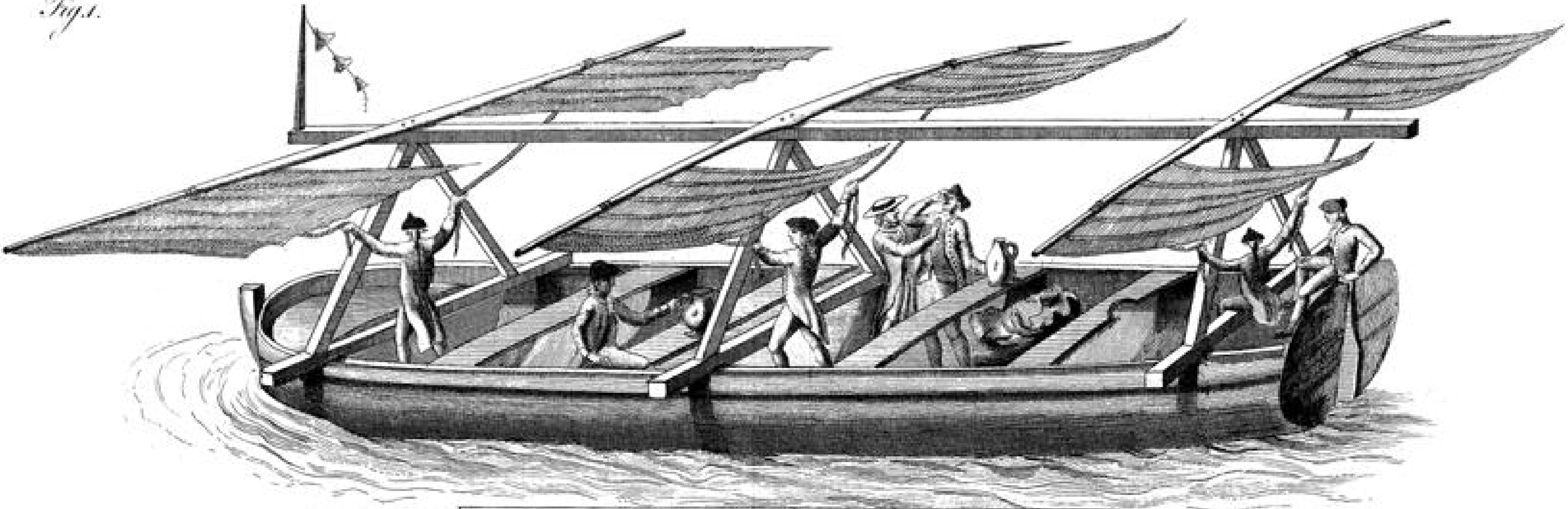
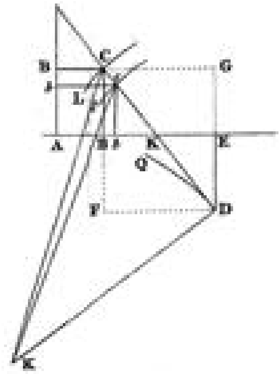


Fig. 2.



Basire S.